The DHC Theorem on Networks and Its Applications

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NetSci June 13 2018 Paris

Abstract

Identifying influential nodes in dynamical processes is crucial in understanding network structure and function. Degree, H-index[1] and coreness[2] are widely used metrics, but previously treated as unrelated. Here we show their relation by constructing an operator $\mathcal{H}$, in terms of which degree, H-index and coreness are the initial, intermediate and steady states of the sequences, respectively[3]. We obtain a family of H-indices that can be used to measure a node's importance. We also prove that the convergence to coreness can be guaranteed even under an asynchronous updating process, allowing a decentralized local method of calculating a node's coreness in large-scale evolving networks. Numerical analyses suggest that the H-index is a good tradeoff that in many cases can better quantify node influence than either degree or coreness. Then we generalized this method to directed and/or weighted networks, and finally we applied this method to the analysis of Brain Networks, online social media influence, and international trade networks, and acquired some new insights.

Model

Recently, H-index was extended to quantify the influence of users in social networks. The H-index of a user v_i is defined as the largest h satisfies that v_i has at least h neighbors for each with a degree no less than h. **Fig. 1** is an explanation of calculating the H-index of a scholar and a user in a social network.

The H-index of a node v_i in a social network can be written as

$$h_i = \mathcal{H}(k_{j_1}, k_{j_2}, \cdots, k_{j_{k_i}})$$

Where $k_{j_1}, k_{j_2}, \cdots, k_{j_{k_i}}$ is the sequence of degree values of v_i's neighbors. The zero-order H-index of v_i is defined as $h_i^{(0)} = k_i$ and the n-order H-index is iteratively defined as

$$h_i^{(n)} = \mathcal{H}(h_{j_1}^{(n-1)}, h_{j_2}^{(n-1)}, \cdots, h_{j_{k_i}}^{(n-1)})$$

Thus, the classical H-index of v_i equals the first-order H-index, i.e., $h_i = h_i^{(1)}$. We proved that after finite steps, the H-indices of every node v_i, say $h_i^{(0)}, h_i^{(1)}, h_i^{(2)}, \ldots$ will converge to its coreness, namely $c_i = h_i^{\infty}$.

Further, we define the zero-order weighted H-index of node v_i as $h_i^{W(0)} = s_i$ and the n-order weighted H-index is iteratively defined as

$$h_i^{w(n)} = \mathcal{H}^w[(w_{ij_1}, h_{j_1}^{w(n-1)}), (w_{ij_2}, h_{j_2}^{w(n-1)}), \cdots, (w_{ij_{k_i}}, h_{j_{k_i}}^{w(n-1)})]$$

where $j_r (r = 1, 2, \ldots, k_i)$ represents v_i's neighbors, whose $(n-1)$-order weighted H-index is $h_{j_r}^{w(n-1)}$ [4].

It can be proved that for every node $v_i \in V$ of a weighted undirected simple network $G(V,E)$, its weighted H-index sequence $h_i^{W(0)}, h_i^{W(1)}, h_i^{W(2)}, \ldots$ will converge to the weighted coreness of node v_i.

It's easy for directed networks, in which the degree of a node is replaced by its in-degree, out-degree, the H_{in}-index and H_{out}-index, in-coreness and out-coreness can also be defined .

THE FAMILY OF H-INDICES OF AN EXAMPLE AND A REAL NETWORKS

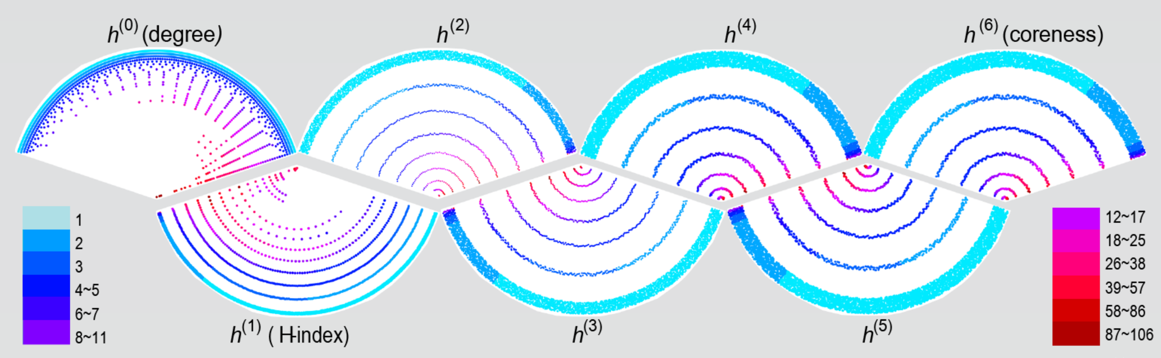

Figure 3. The H-indices of Router network. The colour represents the node degree (from 1 to 106). The node location represents the $h^{(n)}$-index. Nodes located at a 'fan' closer to the centre have higher $h^{(n)}$ values, and nodes located at the same layer have the same $h^{(n)}$ values.

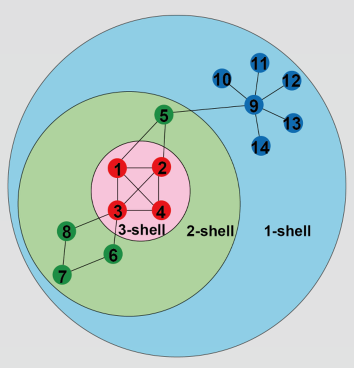

Figure 2. An example network.

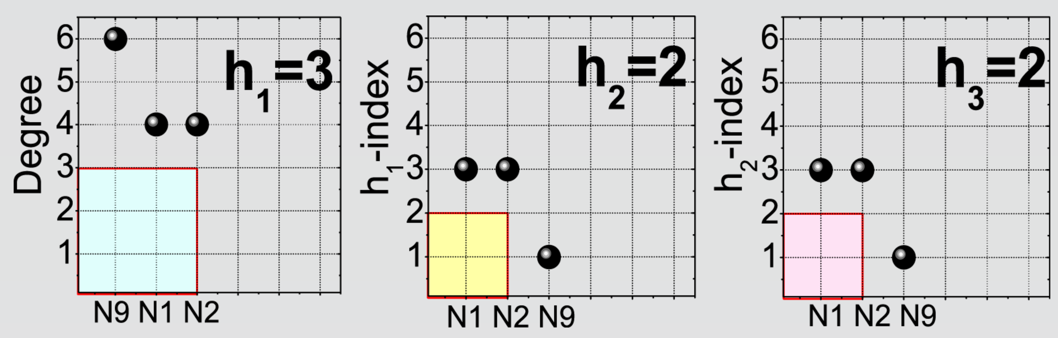

Figure 4. The $h_i^{(1)}$, $h_i^{(2)}$ and $h_i^{(3)}$ of node 5 in Figure 2.

H-INDEX OF A NODE

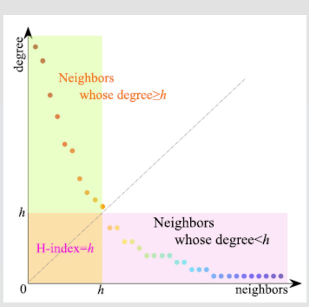

Figure 1. Explanation of calculating the H-index a user in a social network.

APP 1. ANALYSIS TO INTERNATIONAL TRADE NETWORK.

Countries	Rank of coreness in 2010 year	Rank of strength in 2010 year	Rank of coreness in 2014 year	Rank of strength in 2014 year
Canada	1	10	1	10
USA	1	1	1	2
China	3	2	3	1
Japan	3	4	5	4
Belgium	5	11	12	12
France	5	5	7	5
Germany	5	3	7	3
China, HongKong SAR	5	6	5	11
Rep. of Korea	5	9	7	8
Mexico	5	13	4	13
Netherlands	5	7	7	7
United Kingdom	5	6	7	6
Italy	13	8	13	9

Figure 5. The rank of coreness and strength of Top 13 Countries in 2010 and 2014 year respectively.

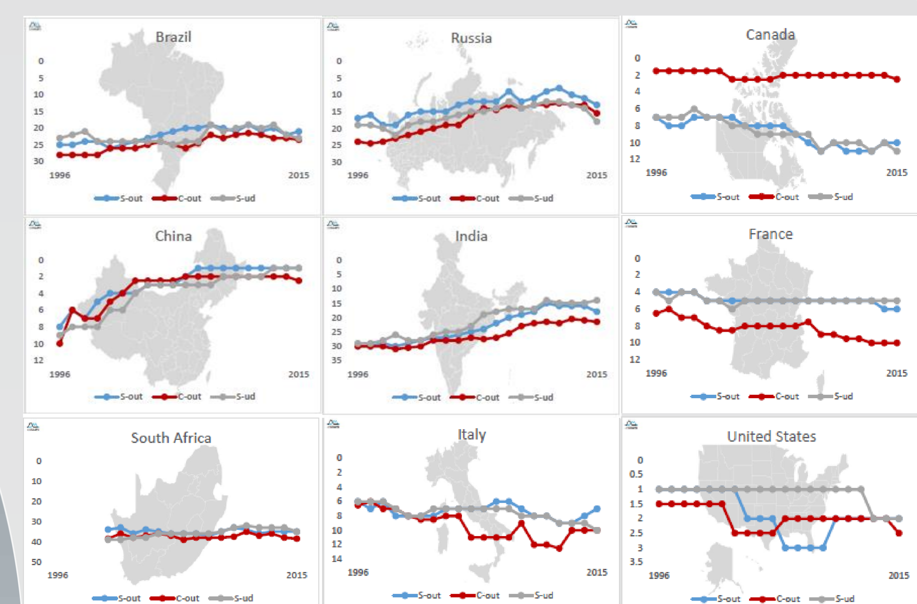

Figure 6. The three indices：out-strength, out-coreness, strength of nine countries from 1996 to 2015 year.

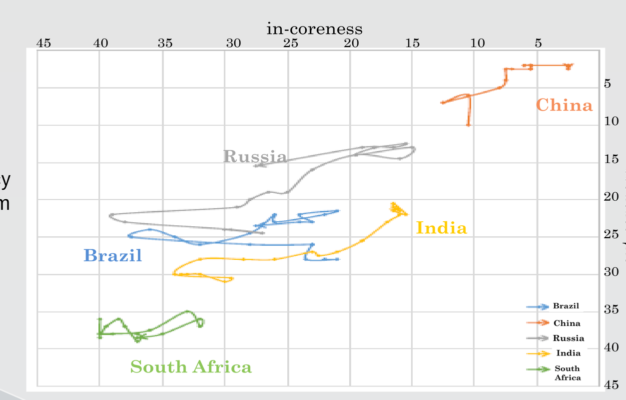

Figure 7. The change tendency of influence of BRICS from 1996 to 2015 year.

APP 3. IDENTIFY INFLUENTIAL CITIES IN ONLINE SOCIAL NETWORKS(WEIBO)

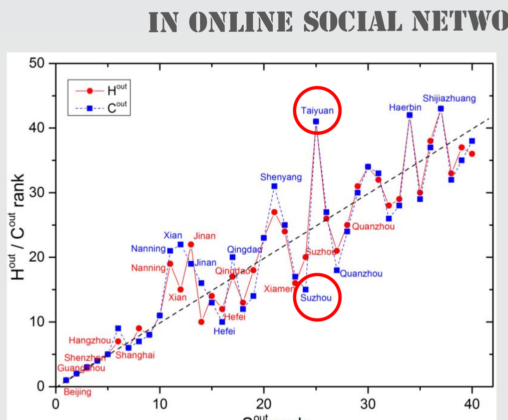

Figure 10. The rank of out-H-index/out-coreness vs. out-strength of cities in China[5].

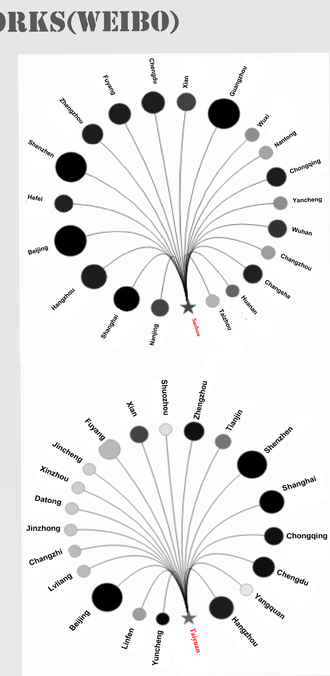

Figure 11. The comparison of influence between Suzhou and Taiyuan.

APP 2. IDENTIFYING AND ANALYZING IMPORTANT BRAIN AREAS FOR DISEASES.

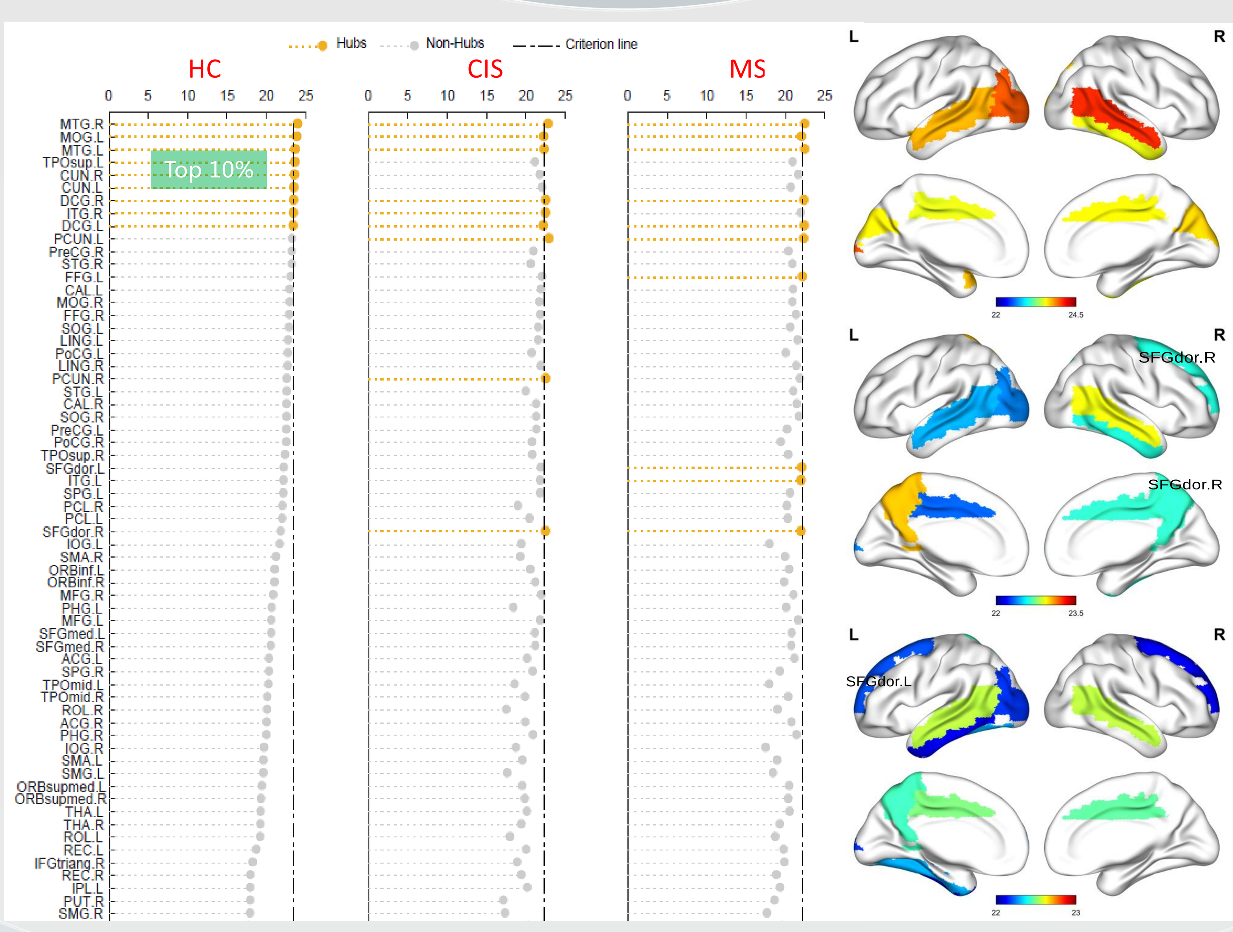

Figure 9. Comparison of the important brain regions of the three groups before and after the illness.

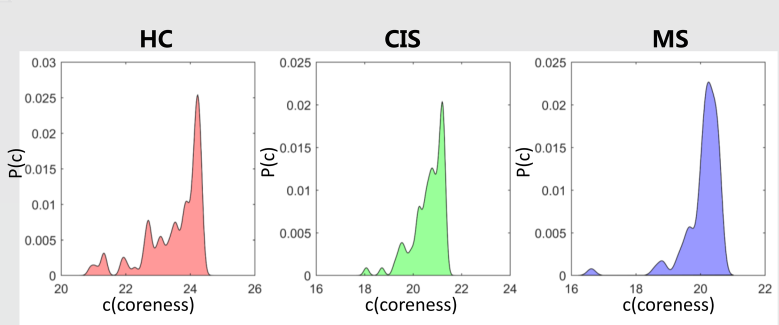

Figure 8. The distribution of coreness of the three groups on the weighted Brain Network.

HC： healthy control
CIS： clinically isolated syndrome
MS： multiple sclerosis

Results

Mathematical relationship.

The DHC Theorem: for every node $i \in V$ of an undirected simple network $G(V, E)$, its H-index sequence $h_i^{(0)}, h_i^{(1)}, h_i^{(2)}, \ldots$ will converge to the coreness of node i,

$$c_i = \lim_{n \to \infty} h_i^{(n)}$$

and then we extended this Theorem to directed and/or weighted networks . We also proved that convergence can still be guaranteed when asynchronous updates are made.

Quantifying spreading influences.

The H-index provides more alternative centralities in characterizing the importance of nodes, and the low-order H-indices are a good tradeoff between degree and coreness. The H-index outperforms both degree and coreness in many social networks. It provides a decentralized calculation method of coreness with higher resolution capability and asynchronous update, and small computational cost, so it can be used for large-scale dynamic networks.

Article

DHC Theorem   Structural Consistency

References

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